Jacobians of curves – Talk 4 – Hilbert schemes, part 2

Goal: For a smooth proper curve C/k, the Hilbert schemes $\underline{Hilb}^d_{C/k} = \underline{H}^d_{C/k}$ are moduli spaces of effective Cartier divisors (eCd) on C. $\forall d \geq 0$.

Recall: For $d \geq 0$, X/S, T/S, define

$$\underline{H}^d_{X/S}(T) = Hilb^d_{X/S}(T) = \{ D \subseteq X_T \text{ closed} \mid D \to S \text{ finite loc free of rank } d \}$$

Rmk: $H^d_{C/k}$ is representable and separated from previous talk, as C/k projective.

Eg: · $H^0_{X/S}$ is always representable with $\underline{H}^0_{X/S} = S$: For T/S,

$$H^0_{X/S}(T) = \{ \emptyset \to X_T \} \xleftrightarrow{1:1} Sch/S(T, S) = \{ T \to S \}$$

· Assume $H^1_{X/S}$ representable, $X \to S$ separated. Then $\underline{H}^1_{X/S} = X$:

Note that finite loc free of rank 1 $\Leftrightarrow$ isomorphism.

As X/S is separated we have $(\Delta_X : X \to X \times_S X) \in H^1_{X/S}(X)$ which is the universal object. We get two maps for T/S

$$H^1_{X/S}(T) \longleftrightarrow Sch/S(T, X)$$
$$f^*(\Delta_X) \longleftarrow\!\shortmid f$$
$$(D \to X_T \to T) \longmapsto (T \to D \to X_T \to X)$$

$$\begin{tikzcd} D \arrow[d] \arrow[dd, bend right=60, "\cong"'] \arrow[dr, bend left] & \\ X_T \arrow[r] \arrow[d] & X \arrow[d] \\ T \arrow[r] & S \end{tikzcd}$$

One can check that they are inverses of each other $\Rightarrow \underline{H}^1_{X/S} = X$

Def: A relative effective Cartier divisor (reCd) on X/S is an eCd $D \subseteq X$ such that $D \to S$ is flat.

Lem: Let $D \hookrightarrow X \to S$ be a reCd. Then for any map $S' \to S$, the pullback $D' \hookrightarrow X' = X \times_S S \to S'$ is an eCd.

$$\begin{tikzcd} D' \arrow[r] \arrow[d] & D \arrow[d] \\ X' \arrow[r, "f'"] \arrow[d] & X \arrow[d] \\ S' \arrow[r, "f"'] & S \end{tikzcd}$$

Note that if $D \leftrightarrow (\mathcal{L}, s)$, then

$$D' \leftrightarrow (f'^*\mathcal{L}, f'^*s)$$

Proof: We have to check that locally f'^*s is a non-zero divisor

→ Wlog, suppose $S = Spec\, R$, $S' = Spec\, R'$, $X = Spec\, A$, $D = V(f) = Spec\, B$.

As f is a non-zero divisor, $(f) \cong A$ as A-modules. We know that B is a flat R-module.

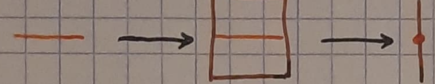
We have a short exact sequence

$$0 \to A \xrightarrow{\cdot f} A \to B \to 0$$

Pulling back by S' is tensoring by R', and we get an exact sequence

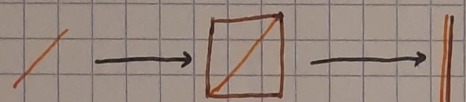
$$\mathrm{Tor}_1^R(B, R') \to A \otimes_R R' \xrightarrow{\cdot f \otimes 1} A \otimes_R R' \to B \otimes_R R' \to 0$$

As B is flat over R, $\mathrm{Tor}_1^R(B, R') = 0$, and $f \otimes 1$ is a non-zero divisor.

Eg: • 

$$k[X] \longleftarrow k[X,Y] \longleftarrow k[Y]$$

The composite $k[Y] \to k[X]$ is not flat as it is not generalizing. Moreover pulling back by the origin gives $\mathbb{A}^1_k \xrightarrow{id} \mathbb{A}^1_k \to \mathrm{Spec}\, k$, and we do not have an eCd.

• 

$$k[X,Y]/(X-Y) \longleftarrow k[X,Y] \longleftarrow k[Y]$$

The composite $k[Y] \to k[X,Y]/(X-Y)$ is an isomorphism and hence is flat.

• If we are working over $S = \mathrm{Spec}\, k$ then eCd $\Leftrightarrow$ reCd, as any morphism $X \to \mathrm{Spec}\, k$ is flat.

Lem: Let $X \to S$ be smooth of rel. dim 1, $D \subseteq X$ closed. Then:

$D \to S$ finite loc. free $\Rightarrow$ $D \hookrightarrow X/S$ reCd.

If $X \to S$ is proper then the converse holds.

Eg: Let $S = \mathrm{Spec}\, k$, X be the affine line with infinitely many origins.

$\Gamma(X, \mathcal{O}_X) = k[T] \Rightarrow V(T)$ is all the origins and is a reCd.

But it is not finite over k. Note that X has a cover by affine lines and hence is smooth of dimension 1 over k.

Rmk: In that setting, $H^d_{X/S}(T)$ is a set of reCd. We can only hope for reCd as pulling back must give another element of the functor.

Rmk: For a smooth proper curve C/k, the Hilbert functors give exactly all the eCd of C, as reCd $\Leftrightarrow$ eCd.

Def: Suppose $X \to S$ smooth of rel dim 1. We say that an eCd $D \subseteq X$ is of degree d if $D \to S$ is finite loc free of rank d, $d \geq 0$.

Def/Prop: Given two eCd $D_1, D_2 \subseteq X$, $D_i \leftrightarrow (\mathcal{L}_i, s_i)$, define their sum $D_1 + D_2$ as the eCd $(\mathcal{L}_1 \otimes_{\mathcal{O}_X} \mathcal{L}_2, s_1 \otimes s_2)$. We have closed immersions $D_i \hookrightarrow D_1 + D_2$. If $D_1, D_2 \subseteq X/S$ reCd, then $D_1 + D_2$ reCd. If X/S smooth of rel dim 1, D_i of degree d_i, then $D_1 + D_2$ of degree $d_1 + d_2$

Proof: We have to check that $s_1 \otimes s_2$ regular section. For an open affine $U = \operatorname{Spec} A \subseteq X$ small enough, $\mathcal{L}_i|_U \cong \tilde{A}$ and

$$\mathcal{L}_1 \otimes_{\mathcal{O}_X} \mathcal{L}_2|_U \xrightarrow{\cong} \tilde{A} \; : \; x \otimes y \longmapsto x \cdot y$$

As s_1, s_2 regular, $s_1|_U, s_2|_U \in A$ are non-zero divisors

$\Rightarrow (s_1 \otimes s_2)|_U = s_1|_U \cdot s_2|_U \in A$ is a non-zero divisor $\Rightarrow s_1 \otimes s_2$ is regular.

Moreover, $D_i \cap U = V_U(s_i|_U)$; $(D_1 + D_2) \cap U = V_U(s_1|_U \cdot s_2|_U)$

$\Rightarrow V_U(s_i|_U) \subseteq V_U(s_1|_U \cdot s_2|_U)$ closed, which glue to a closed immersion $D_i \hookrightarrow D_1 + D_2$. The other cases are omitted.

Eg: Consider $\mathbb{A}^1_{\mathbb{C}} \to \operatorname{Spec} \mathbb{C}$. For $z \in \mathbb{C}$ define $D_z = \operatorname{Spec} \mathbb{C}[X]/(X - z) \hookrightarrow \mathbb{A}^1_{\mathbb{C}}$.

$\to D_z$ is an eCd of degree 1 over $\mathbb{C}$. Take $z \neq z'$. We get

$D_z + D_z = \operatorname{Spec} \mathbb{C}[X]/(X-z)^2$ eCd of degree 2.

$D_z + D_{z'} = \operatorname{Spec} \mathbb{C}[X]/(X-z)(X-z') = D_z \amalg D_{z'}$

As $\mathbb{C}$ is algebraically closed we can write any non-zero polynomial as

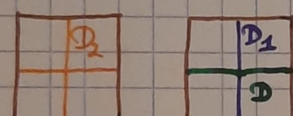
$$f = (X - z_1)^{d_1} \cdot \ldots \cdot (X - z_n)^{d_n}; \qquad d_i \geq 1, \; z_i \in \mathbb{C}.$$

$\Rightarrow V(f) = d_1 \cdot D_{z_1} + \ldots + d_n \cdot D_{z_n}$ eCd of degree $d_1 + \ldots + d_n$.

$\to$ An eCd of degree d corresponds to choosing d closed points of $\mathbb{A}^1_{\mathbb{C}}$ with repetitions allowed.

Lem: If $D_1, D_2 \subseteq X$ eCd, $D_1 \subseteq D_2$ closed, then there exists a unique eCd $D \subseteq X$ with $D_2 = D_1 + D$. If $D_1, D_2 \subseteq X/S$ relCd, then D relCd. If $X \to S$ smooth of rel dim 1, D_i of degree d_i, then D of degree $d_2 - d_1$.

Proof: Write $D_i \leftrightarrow (\mathcal{L}_i, s_i)$ and let $\mathcal{L} := \mathcal{L}_2 \otimes_{\mathcal{O}_X} \mathcal{L}_1^\vee \Rightarrow \mathcal{L}_1 \otimes_{\mathcal{O}_X} \mathcal{L} = \mathcal{L}_2$.
For an open affine $U = \operatorname{Spec} A \subseteq X$ small enough, all these line bundles are trivial. As $D_1 \cap U \subseteq D_2 \cap U$, we have $(s_1|_U) \supseteq (s_2|_U)$ as ideals in A.
$\Rightarrow$ There exists $t|_U \in A$ with $s_1|_U \cdot t|_U = s_2|_U$. As $s_1|_U, s_2|_U$ are non-zero divisors, so is $t|_U$. For any other $t' \in A$ with $s_1|_U t' = s_2|_U$, we have $0 = s_1|_U (t|_U - t') \Rightarrow t|_U = t'$ as $s_1|_U$ non-zero divisor.
Hence for any such U we get a unique section $t|_U \in \mathcal{L}(U)$ with $t|_U \otimes s_1|_U = s_2|_U \in \mathcal{L}_2(U)$. By uniqueness they glue to a global regular section $t \in \mathcal{L}(X)$. We can take $D \leftrightarrow (\mathcal{L}, t)$.
The other cases are omitted.

Eg: 

Take $X = \mathbb{A}^2_k$, $D_2 = V(XY)$, $D_1 = V(X)$
$\rightarrow D = V(Y)$

- Consider $X = \mathbb{A}^1_{\mathbb{C}}$, $D_2 = V((X-z_1)^{d_1} \cdot \ldots \cdot (X-z_n)^{d_n})$,
$D_1 = V((X-z_1)^{d_1'} \cdot \ldots \cdot (X-z_n)^{d_n'})$ $\quad d_i' \leq d_i$
$\rightarrow D = V((X-z_1)^{d_1 - d_1'} \cdot \ldots \cdot (X-z_n)^{d_n - d_n'})$

This addition gives a natural transformation (From now on $X \to S$ smooth of rel dim 1):
$$H^{d_1}_{X/S} \times H^{d_2}_{X/S} \longrightarrow H^{d_1+d_2}_{X/S} : (D, D') \longmapsto D + D'$$
which translates to a morphism of schemes if they are representable, which we will assume from now on. Taking $d_2 = 1$ gives a map
$$\underline{H}^d_{X/S} \times_S X \longrightarrow \underline{H}^{d+1}_{X/S} : (D, x) \longmapsto D + x$$

Lem: This map is finite locally free of rank $d+1$.
Proof: Let $D_U \subseteq \underline{H}^{d+1}_{X/S} \times_S X$ be the universal object of $\underline{H}^{d+1}_{X/S}$.
We want to understand its functor of points.

We have a bijection for T/S

$$\mathrm{Sch}/S(T, D_u) \xleftrightarrow{1:1} \{(D, x) \in H^{d+1}_{X/S} \times H^1_{X/S}(T) \mid x \subseteq D \text{ closed}\}$$

$$(f: T \to D_u) \longmapsto (D, x) \text{ coming from } T \to D_u \to \underline{H}^{d+1}_{X/S} \times_S X$$

$$(T \to x \to D \to D_u) \longleftarrow\!\shortmid (D, x)$$

Given a map $T \to D_u$ we get a diagram as on the left. The composite

$$T \to D_u \to \underline{H}^{d+1}_{X/S} \times_S X$$

give us D and x. As D_u is the universal object of $\underline{H}^{d+1}_{X/S}$, the upper square is a pullback.

$$\begin{tikzcd}
 & D \arrow[rr] \arrow[d] \arrow[drr, phantom, "\lrcorner"] & & D_u \arrow[d] & & \\
x \arrow[ur] \arrow[r] \arrow[dr, "\cong"'] & X_T \arrow[rr] \arrow[d] \arrow[drr, phantom, "\lrcorner"] & & \underline{H}^{d+1}_{X/S} \times_S X \arrow[rr] \arrow[d] \arrow[drr, phantom, "\lrcorner"] & & X \arrow[d] \\
 & T \arrow[rr] \arrow[uurr] & & \underline{H}^{d+1}_{X/S} \arrow[rr] & & S
\end{tikzcd}$$

As x has maps into X_T and D_u by UP of pullback we get a map $x \to D$. Now $x \to X_T$ and $D \to X_T$ are closed immersions, hence $x \to D$ is one as well. Conversely given a closed immersion $x \to D$ we get a map $T \to D_u$ as the composite

$$T \to x \to D \to D_u,$$ because $x \to T$ is an isomorphism.

The morphism we are interested in is the composite

$$\underline{H}^d_{X/S} \times_S X \longrightarrow \underline{H}^{d+1}_{X/S} \times_S X \longrightarrow \underline{H}^{d+1}_{X/S}$$

$$(D', x) \longmapsto (D' + x, x)$$

By the previous discussion this factors as in the diagram.

$$\begin{tikzcd}
(D', x) \arrow[rr, mapsto] & & (D' + x, x) \\
\underline{H}^d_{X/S} \times_S X \arrow[rr] \arrow[dr] & & D_u \arrow[dl] \\
 & \underline{H}^{d+1}_{X/S} &
\end{tikzcd}$$

Given $(x \subseteq D) \in D_u(T)$ by the previous lemma there is a unique $D' \in \underline{H}^d_{X/S}(T)$ with $D' + x = D$. Hence the top map has an inverse $(D, x) \longmapsto (D', x)$ and is an isomorphism.

By definition $D_u \longrightarrow \underline{H}^{d+1}_{X/S}$ is finite loc free of degree $d+1$, which concludes.

Lem: $\underline{H}^d_{X/S} \to S$ is smooth of relative dimension d.

Proof: It is true for $d=0,1$ as we know the schemes. Suppose it holds for $d-1$. We get the diagram on the left where the top map is finite loc free and the left one is smooth of rel. dim. d.

$$\begin{tikzcd} \underline{H}^{d-1}_{X/S} \times_S X \arrow[rr] \arrow[dr] & & \underline{H}^d_{X/S} \arrow[dl] \\ & S & \end{tikzcd}$$

A complicated lemma allows to conclude in this case.

Lem: If $f: X/S \to Y/S$ surjective, $p: X \to S$ (universally) closed, then $q: Y \to S$ is (universally) closed.

Proof: Suppose p closed and take a closed subscheme $T \subseteq Y$. Then

$$p(f^{-1}(T)) = q(f(f^{-1}(T))) \overset{f \text{ surjective}}{=} q(T) \quad \text{is closed.}$$

As the hypothesis are closed under base change the universal case follows.

Def: X/k is geometrically irreducible if $X_{k'}$ is irreducible for all field extensions k'/k.

Lem: If X/k irreducible, k alg. closed, then X/k geometrically irreducible.

Eg: • The vanishing locus $V(f) \subseteq \mathbb{A}^n_k$ of a polynomial that is irreducible over k but not its algebraic closure is not geometrically irreducible.

→ Take $V((X+Y)^2+1) \subseteq \mathbb{A}^2_{\mathbb{R}}$. Base changing over $\mathbb{C}$ gives

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}[X,Y]/((X+Y)^2+1) = \mathbb{C}[X,Y]/(X+Y+i)(X+Y-i)$$
$$\cong \mathbb{A}^1_{\mathbb{C}} \amalg \mathbb{A}^1_{\mathbb{C}}$$

→ $V(Y^2+X^2+1) \subseteq \mathbb{A}^2_{\mathbb{R}}$. As Y^2+X^2+1 is irreducible over $\mathbb{C}$, base changing gives $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}[X,Y]/(Y^2+X^2+1) = \mathbb{C}[X,Y]/(Y^2+X^2+1)$ which is still irreducible.

Lem: If X/k is geometrically irreducible, then $X^d = X \times_k \dots \times_k X$ is irreducible.

Th: Let C/k be a geometrically smooth proper curve over k. Then:

1) $H^d_{C/k}$ is representable by a smooth proper variety over k $\forall d \geq 0$.

2) For a field extension k'/k, $\underline{H}^d_{C/k}(k')$ corresponds exactly to the set of eCd of degree d on $C_{k'}$.

3) We have a surjection $C^d \longrightarrow \underline{H}^d_{C/k}$ finite loc free of rank $d!$.

$(x_1, \dots, x_d) \longmapsto x_1 + \dots + x_d$

For $d_1, d_2 \geq 0$ we have a finite loc free of rank $\binom{d_1+d_2}{d_1}$ map

$$\underline{H}^{d_1}_{C/k} \times_k \underline{H}^{d_2}_{C/k} \longrightarrow \underline{H}^{d_1+d_2}_{C/k} : (D, D') \longmapsto D + D'$$

Proof: 1) We know that $H^d_{C/k}$ is representable, separated and smooth. By induction we get a surjection $C^d \to \underline{H}^d_{C/k} : (x_1, \dots, x_d) \mapsto x_1 + \dots + x_d$ which is finite loc free of rank $d!$. Now C^d is irreducible and that map is surjective $\Rightarrow$ $\underline{H}^d_{C/k}$ is irreducible. Moreover $C^d \to k$ universally closed $\Rightarrow$ $\underline{H}^d_{C/k} \to k$ universally closed, and hence proper.

2) By definition of $H^d_{C/k}$ and the discussion about divisors.

3) We already know the map $C^d \to \underline{H}^d_{C/k}$. For $d_1, d_2 \geq 0$ we have:

$$\begin{tikzcd} C^{d_1} \times_k C^{d_2} \arrow[r, "\cong"] \arrow[d] & C^{d_1+d_2} \arrow[d] \\ \underline{H}^{d_1}_{C/k} \times_k \underline{H}^{d_2}_{C/k} \arrow[r] & \underline{H}^{d_1+d_2}_{C/k} \end{tikzcd}$$

The left vertical map is finite loc free of rank $d_1! \cdot d_2!$, and the right vertical map of rank $(d_1+d_2)!$

$\Rightarrow$ the lower map is finite loc free of rank

$$\frac{(d_1+d_2)!}{d_1! \cdot d_2!} = \binom{d_1+d_2}{d_1}$$

The map $C^d \longrightarrow \underline{H}^d_{C/k}$ is finite loc. free of rank $d!$ and does not depend on the order of the arguments. Therefore we would expect $\underline{H}^d_{C/k}$ to be exactly "C^d/S_d", where S_d is the symmetric group on d elements.

This will be the topic of the next talk.